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Quasi-exactly solvable quartic Bose Hamiltonians

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Abstract

We consider Hamiltonians, which are even polynomials of the fourth order with respect to Bose operators. We find subspaces, preserved by the action of the Hamiltonian. These subspaces, being finite dimensional, include, nonetheless, states with an *infinite* number of quasi-particles, corresponding to the original Bose operators. The basis functions look rather simple in the coherent state representation and are expressed in terms of the degenerate hypergeometric function with respect to the complex variable labelling the representation. In some particular degenerate cases they turn (up to the power factor) into trigonometric or hyperbolic functions, Bessel functions or combinations of the exponent and Hermite polynomials. We find explicitly the relationship between coefficients at different powers of Bose operators that ensure quasi-exact solvability of Hamiltonians.

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1. Introduction

The conception of quasi-exactly solvable (QES) systems, discovered in the 1980s [1–12], has received much attention in recent years, both from the viewpoint of physical applications and their inner mathematical beauty. It turned out that in quantum mechanics there exists a peculiar class of objects that occupy an intermediate place between exactly solvable and non-solvable models, in the sense that in an infinite Hilbert space a finite part of a spectrum is singled out within which eigenvectors and eigenvalues can be found from an algebraic equation of a finite degree—in other words, a partial algebraization of the spectrum occurs. For one-dimensional QES models the corresponding QES Hamiltonians possess hidden group structure based on $sl(2, R)$ algebra. Thus, they have direct physical meaning, being related to quantum spin systems [13].

Meanwhile, the notion of QES systems is not constrained by potential models and can have a nontrivial meaning for any kind of infinite-dimensional systems. In the first place, it concerns Bose Hamiltonians whose physical importance is beyond doubt. Here one should distinguish two cases. First, it turns out that some systems of two interacting particles or quasi-particles with Bose operators of creation and annihilation a, a^+ and b, b^+ can be mapped on the problem for a particle moving in a certain type of one-dimensional potential and, remarkably, these potentials belong just to the QES type [14–16]. In particular, such a type of Hamiltonian is widespread in quantum optics and the physics of magnetism [13]. The aforementioned mapping works only for a special class of Bose Hamiltonians which possess an integral of motion. Then the procedure is performed in three steps: (i) all Hilbert space splits in a natural way into different pieces with respect to the values of an integral of motion, (ii) in each piece the Schrödinger equation takes a finite-difference form, (iii) it is transformed into the differential equation by means of introducing a generating function. In so doing, the integral of motion under discussion represents a linear combination of numbers of particles a^+a and b^+b.

The second kind of Bose systems looks much more usual—it is simply some polynomial with respect to the Bose operators of creation and annihilation of one particle. The fact that only one pair a, a^+ enters the Hamiltonian deprives us, in contrast to the first case, of the possibility to construct a simple integral of motion—in this sense the eigenvalue problem becomes more complicated. In general, the solutions of the Schrödinger equation contain infinite numbers of quasi-particles and only approximate or numerical methods can be applied to such systems. However, as was shown recently [17], if the coefficients at different powers of a, a^+ are selected in a proper way, in some cases a *finite-dimensional* closed subspace is singled out and algebraization of the spectrum occurs similar to what happens in the 'usual' QES potential models or differential equations. In so doing, the eigenvectors belonging to the subspace under discussion can be expressed as a finite linear combination of eigenvectors of an harmonic oscillator and, thus, contain a finite number of quasi-particles [17].

In this paper we extend the approach of [17] and consider much more general classes of Hamiltonians. Their distinctive feature consists in that the relevant basis functions that compose a finite-dimensional subspace look very much unlike the wavefunctions of a harmonic oscillator. As a result, we obtain QES models with an *infinite* numbers of quasi-particles in this *finite*-dimensional subspace. Bearing in mind the physical application, we make an emphasis on Hermitian Hamiltonians, although our approach is applicable to more general QES Bose operators without the demand of hermiticity.

2. Bose Hamiltonians as differential operators and structure of invariant subspaces

Consider the operator which is the even polynomial of the fourth degree with respect to the Bose operators of creation a^+ and annihilation a. It can be written in the form

$$H = a_{++}K_+^2 + a_{--}K_-^2 + a_{00}K_0^2 + a_{0-}K_0K_- + a_{+0}K_+K_0 + a_0K_0 + a_-K_- + a_+K_+ \tag{1}$$

where

$$K_0 = \tfrac{1}{2}(a^+a + \tfrac{1}{2}) \qquad K_- = \frac{a^2}{2} \qquad K_+ = \frac{a^{+2}}{2} \tag{2}$$

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2K_0. \tag{3}$$

The Casimir operator is $C = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+) \equiv -\frac{3}{16}$.

We will use the coherent state representation in which

$$a \to \frac{\partial}{\partial z} \qquad a^+ \to z. \tag{4}$$

After substitution in (1) the Hamiltonian $H(a^+, a)$ becomes a differential operator $H(z, \frac{\partial}{\partial z})$. In the previous Letter [17] we discussed Bose systems that possess the invariant subspace of the form $F = \mathrm{span}\{z^n\}$ or $\mathrm{span}\{z^{2n}\}$. The first natural step towards generalization consists in considering subspaces (with N fixed)

$$F = \mathrm{span}\{u_n\} \qquad u_n = z^{2n}u \qquad n = 0, 1, 2 \ldots N \tag{5}$$

for which the following procedure should be realized: (i) the action of operators of K_i on the functions u_n should lead to the linear combination of functions from the same set $\{u_n\}$, (ii) by the selection of appropriate coefficients in (1), we find the subspace F to be closed under the action of the Hamiltonian H. We would like to stress that condition (i) does not forbid u_n with $n > N$ to appear in terms like $K_i u_n$ but condition (ii) rules out such functions from Hu_n (recall that we consider Hamiltonians which are quadratic-linear combinations of K_i).

It is seen from (2) and (4) that the operators K_i contain z and $\frac{\partial}{\partial z}$. Therefore, it is convenient to assume that differentiation of $u(z)$ gives rise to u up to the factor that contains powers of z. The corresponding choice is not unique. In this paper we restrict ourselves to one of the simplest possibilities that leads to nontrivial solutions. To this end, we choose u that obeys the differential equation

$$u' = A(z)u \qquad A(z) = \left(\frac{\beta}{z} + 2\rho z\right). \tag{6}$$

We will show below that the choice (6) relates $K_i u_n$ to u_n, $u_{n\pm1}$ that, in turn, allows us to formulate the conditions of the cut off for the Hamiltonian in the form of algebraic equations which its coefficients obey. It follows from (6) that $u = z^\beta \exp(\rho z^2)$. To ensure asymptotic analytic behaviour near $z = 0$, we demand that $\beta = 0, 1, 2 \ldots.$ Now let us take into account some basic properties of coherent states (see, e.g. Ch. 7 of [18]). Our functions $u_n(z)$ must belong to the Bargmann–Fock space. It means that they should obey the conditions of integrability and analyticity. The condition of integrability for any two functions f, g from our space

$$\int \mathrm{d}z\, \mathrm{d}z^* f^* g \mathrm{e}^{-zz^*} < \infty \tag{7}$$

entails, for our choice of u, $|\rho| < 1/2$.

Taking into account equation (6), it is straightforward to show that

$$K_+u_n = C_+u_{n+1} \tag{8}$$

$$K_-u_n = A_-(n)u_n + B_-(n)u_{n-1} + C_-u_{n+1} \tag{9}$$

$$K_0u_n = A_0(n)u_n + C_0u_{n+1} \tag{10}$$

where $C_+ = \frac{1}{2}$, $C_- = 2\rho^2$, $C_0 = \rho$, $A_-(n) = (2\beta + 4n + 1)\rho$, $A_0(n) = \frac{2\beta+4n+1}{4}$, $B_-(n) = \frac{(\beta+2n)(\beta+2n-1)}{2}$.

Using equations (8)–(10), one can present the action of the operator (1) in the form

$$Hu_n = D_2u_{n+2} + D_1(n)u_{n+1} + \tilde{D}_0(n)u_n + \tilde{D}_1(n)u_{n-1} + \tilde{D}_2(n)u_{n-2} \tag{11}$$

where

$$D_2 = \frac{a_{+0}}{2}C_0 + a_{0-}C_-C_0 + a_{00}C_0^2 + a_{--}C_-C_- + \frac{a_{++}}{4} \tag{12}$$

$$\begin{aligned} D_1(n) = {} & a_{--}\left[A_-(n)C_- + C_-A_-(n+1)\right] + a_{00}\left[A_0(n)C_0 + C_0A_0(n+1)\right] + \frac{a_{+0}}{2}A_0(n) \\ & + a_{0-}[A_-(n)C_0 + C_-A_0(n+1)] + a_0C_0 + a_-C_- + \frac{a_+}{2} \end{aligned} \tag{13}$$

$$\tilde{D}_0 = a_{00}A_0^2(n) + a_{0-}[A_-(n)A_0(n) + B_-(n)C_0] + a_0A_0(n) + a_-A_-(n) \tag{14}$$

$$\tilde{D}_1(n) = a_{--}\left[A_-(n)B_-(n) + B_-(n)A_-(n-1)\right] + a_{0-}B_-(n)A_0(n-1) + a_-B_-(n) \tag{15}$$

$$\tilde{D}_2 = a_{--}B_-(n)B_-(n-1). \tag{16}$$

For the operator (1) to be QES with the invariant subspace (5), it is necessary that the following conditions of cut off be satisfied:

$$D_2 = 0 \tag{17}$$

$$D_1(N) = 0 \tag{18}$$

$$\tilde{D}_1(0) = 0 \tag{19}$$

$$\tilde{D}_2(0) = 0 \tag{20}$$

$$\tilde{D}_2(1) = 0. \tag{21}$$

In general, this system is rather cumbersome. However, it is simplified greatly if we consider Hermitian Hamiltonians with $a_{--} = 0 = a_{++}$. Then $\tilde{D}_2 \equiv 0$ and we get three equations:

$$\rho^2 a_{00} + 2\rho^3 a_{0-} + \tfrac{1}{2}\rho a_{0-} = 0 \tag{22}$$

$$\left[\frac{a_{0-}}{4}(2\beta - 3) + a_-\right]\beta(\beta - 1) = 0 \tag{23}$$

$$\frac{1+4\rho^2}{2}a_- + \rho a_0 + \left[\frac{2\beta + 4N + 1}{8} + \rho^2\frac{7 + 6\beta + 12N}{2}\right]a_{0-} + \frac{a_{00}}{2}\rho(2\beta + 4N + 3) = 0. \tag{24}$$

It is assumed that $a_+ = a_-$, $a_{+0} = a_{0-}$ and all coefficients are real. The analysis leads to the following table of possible solutions:

$a_{++} = 0$

	ρ	β	a_-	a_{00}	a_0
1	0	0	$-\frac{4N+1}{4}a_{0-}$	a.v.	a.v.
2	0	1	$-\frac{4N+3}{4}a_{0-}$	a.v.	a.v.
3	a.v.	a.v.	$\frac{3-2\beta}{4}a_{0-}$	$f(\rho)a_{0-}$	$\left[\frac{2\beta+2N+1}{4\rho} - 2(N+1)\rho\right]a_{0-}$

$a_{++} \neq 0$

4	a.v.	0	a.v.	$f(\rho)a_{0-}$	$f(\rho)a_- + f_1(\rho, N)a_{0-} + f_2(\rho, N)a_{++}$
5	a.v.	1	a.v.	$f(\rho)a_{0-}$	$f(\rho)a_- + f_1\left(\rho, N + \frac{1}{2}\right)a_{0-} + f_2\left(\rho, N + \frac{1}{2}\right)a_{++}$

Here 'a.v.' denotes 'arbitrary value', with the reservation that β is a positive integer or zero and $|\rho| < 1/2$, as is explained above. By definition

$$f(\rho) = -\frac{1+4\rho^2}{2\rho} \qquad f_1(\rho, N) = \frac{4N+5}{8\rho} - \frac{4N+1}{2}\rho$$
$$f_2(\rho, N) = -\frac{4N+3}{8\rho^2}(16\rho^4 - 1). \tag{25}$$

One can check that solution 1, when $u = 1$, corresponds to even states for the example considered in equation (7) of [17], provided the coefficient $A_2 = 0$ in that equation. In a similar way, case 2 ($u = z$) corresponds to odd states from the same example. However, cases (3)–(5) represent new solutions that were not contained in [17].

3. Doubled invariant subspaces

A much richer family of new classes of Bose QES quartic Hamiltonians can be obtained if we generalize the structure of the invariant subspace introducing, in addition to u_n, a second subset of independent functions. Consider the set of functions

$$u_n = z^{2n}u \qquad v_n = z^{2n+1}v \qquad n = 0, 1, 2 \ldots. \tag{26}$$

We are interested in such a function which forms a set, defined for a fixed $N = 0, 1, 2, \ldots$:

$$F = \mathrm{span}\,\{u_n, v_n\} = \mathrm{span}\left\{z^{2n} \cdot u\,(z)\,, z^{2n+1} \cdot v\,(z)\,\right\} \qquad n = 0, 1, 2, \ldots, N \tag{27}$$

invariant with respect to the action of the operator H. The dimension of F is equal to $2(N+1)$. It may happen that, for some values of the parameters, the functions v_n may be proportional or even exactly equal to u_n. Then our subspace reduces to the $N+1$ subspace (5) considered in a previous section. To gain qualitatively new QES models, in what follows we will consider the functions u_n and v_n as, generally speaking, independent.

Let u and v obey the system of differential equations that generalizes the relation (6):

$$u' = Au + Bv \tag{28}$$

$$v' = Cu + Dv. \tag{29}$$

Here the prime denotes differentiation with respect to z, and A, B, C, D are functions of z.

It follows from (28) and (29) that

$$u'' - u'\left(S + \frac{B'}{B}\right) + u\left[\Delta + \frac{W(B, A)}{B}\right] = 0 \tag{30}$$

$$v'' - v'\left(S + \frac{C'}{C}\right) + v\left[\Delta + \frac{W(C, D)}{C}\right] = 0. \tag{31}$$

Here $S = A + D = SpL$, where $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $W(f_1, f_2) \equiv f_1' f_2 - f_1 f_2'$ is a Wronskian and $\Delta = AD - BC$ is a determinant of L. In what follows we assume for simplicity the quantities in the denominators in (30) and (31) are $B(z) \equiv \alpha = \mathrm{const}$ and $C(z) = \gamma = \mathrm{const}$. Then we have

$$u'' - u'S + u(\Delta - A') = 0 \tag{32}$$

$$v'' - v'S + v(\Delta - D') = 0. \tag{33}$$

We assume also, by analogy with (6), that A and D contain only terms of the order z and z^{-1} in the Loran series: $A = 2\rho z + \beta z^{-1}$, $D = 2\tau z + \delta z^{-1}$. Then we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}u\,(z) &= \alpha v\,(z) + \frac{\beta}{z}u\,(z) + 2\rho z u\,(z) \\ \frac{\mathrm{d}}{\mathrm{d}z}v\,(z) &= \gamma u\,(z) + \frac{\delta}{z}v\,(z) + 2\tau z v\,(z)\,. \end{aligned} \tag{34}$$

One obtains from (32) and (33)

$$u'' - u'\left[\frac{\delta + \beta}{z} + 2(\rho + \tau)z\right] + u\left[\frac{\beta(1 + \delta)}{z^2} + 4\rho\tau z^2 + 2\beta\tau + 2\rho\delta - 2\rho - \alpha\gamma\right] = 0 \tag{35}$$

$$v'' - v'\left[\frac{\delta + \beta}{z} + 2(\rho + \tau)z\right] + v\left[\frac{\delta(1 + \beta)}{z^2} + 4\rho\tau z^2 + 2\tau\beta + 2\rho\delta - 2\tau - \alpha\gamma\right] = 0. \tag{36}$$

Our functions $u_n(z)$ and $v_n(z)$ must belong to the Bargmann–Fock space that entails, similarly to what is obtained in the previous section, the conditions $|\rho| < 1/2$, $|\tau| < 1/2$.

To elucidate what constraints are imposed by the demand of analyticity, consider separately several different cases. If $\gamma = \alpha = 0$, equations (34) can be integrated and one easily finds that $u = z^{\beta}\exp(\rho z^2)$, $v = z^{\delta}\exp(\tau z^2)$, whence it is obvious that $\beta = 0, 1, 2\ldots$ and $\delta = -1, 0, 1, 2\ldots$. If $\alpha = 0$ but $\gamma \neq 0$, one can make the substitution $v = z^{\delta}\exp(\tau z^2)w$. Then

$$w' = \gamma z^{\beta-\delta}\exp[(\rho - \tau)z^2]. \tag{37}$$

It is clear that $\beta = 0, 1, 2, \ldots$, whereas δ is arbitrary except for $\delta = \beta + 1$, since the latter would have led to the logarithmic terms in $v(z)$. The similar situation occurs when $\gamma = 0$ but $\alpha \neq 0$. Then $\delta = -1, 0, 1, 2\ldots$ and forbidden values of β are $\beta = \delta + 1$.

Now let $\alpha\gamma \neq 0$. First, consider the case $\rho \neq \tau$. Then by substitutions

$$u(z) = y(z^2(\tau - \rho)) \cdot \exp(\rho z^2) \cdot z^{\beta} \tag{38}$$

$$v(z) = \tilde{y}(z^2(\rho - \tau)) \cdot \exp(\tau z^2) \cdot z^{\delta} \tag{39}$$

equations (35) and (36) are reduced to the form, typical of a degenerate hypergeometric function,

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) + (\eta - x)\frac{\mathrm{d}}{\mathrm{d}x}y(x) - \xi y(x) = 0 \tag{40}$$

where $\eta = \frac{1}{2}(\beta - \delta + 1)$, $\xi = \frac{\alpha\gamma}{4(\tau-\rho)}$. The function $\tilde{y}$ satisfies the equation of the same form (40) but with parameters $\tilde{\eta} = 1 - \eta$, $\tilde{\xi} = -\xi$.

To determine the admissible range of parameters β, δ one can appeal directly to the well-known properties of this function and take into account that the general solution of equation (40) has the form $y = Ay_1 + By_2$, where $y_1 = \Phi(\xi, \eta; x)$ and $y_2 = x^{1-\eta}\Phi(\xi - \eta + 1, 2 - \eta; x)$ and the standard notation for the degenerate hypergeometric function is used (see Ch. 6 of [19]). First, consider the case when η is non-integer. Then $\Phi \to 1$ when $x \to 0$ and from (38) we obtain the function u, which can have two possible asymptotic forms: $u_1 \sim z^{\beta}$ and $u_2 \sim z^{\delta+1}$. The function $v(z)$ behaves, correspondingly, like $v_1 \sim z^{\beta+1}$ and $v_2 \sim z^{\delta}$. Therefore, it turns out that there are two cases:

$$\beta = 0, 1, 2, \ldots, \delta \text{ is arbitrary} \tag{41}$$

or

$$\delta = -1, 0, 2, \ldots, \beta \text{ is arbitrary.} \tag{42}$$

If η is integer, there exists only one independent solution of equation (40), regular at $x \to 0$. The corresponding solution is known to be $\Phi^*(\xi, \eta; x) = \frac{\Phi(\xi,\eta;x)}{\Gamma(\eta)}$. In the limit $x \to 0$ $\Phi^* \sim x^{1-\eta}$ this does not affect the conclusion about the admissible range of β and δ.

Now let $\rho = \tau$, $\alpha\gamma \neq 0$. By substitution

$$u(z) = y\left(2\sqrt{\alpha\gamma}z\right) \cdot \exp\left(z\left(\rho z - \sqrt{\alpha\gamma}\right)\right) \cdot z^{\beta}$$

we find that the function $y(x)$ obeys the equation

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) + (\beta - \delta - x)\frac{\mathrm{d}}{\mathrm{d}x}y(x) - \frac{(\beta - \delta)}{2}y(x) = 0$$

which has the same form as (40) and admissible β and δ satisfy one of the criteria (41) or (42).

Differential equations for our functions can also be written in the symmetric form. Let us make the substitution

$$u = \Psi z^{\frac{\beta+\delta}{2}}\exp\left[\frac{(\rho + \tau)}{2}z^2\right]. \tag{43}$$

Then

$$\Psi'' + (\varepsilon - V_{\text{eff}})\Psi = 0 \tag{44}$$

where

$$V_{\text{eff}} = \frac{k}{z^2} + (\rho - \tau)^2 z^2 \tag{45}$$

$$k = \frac{(\delta - \beta)(\delta - \beta + 2)}{4} \tag{46}$$

$$\varepsilon = (\rho - \tau)(\delta - \beta - 1) - \alpha\gamma. \tag{47}$$

In a similar way

$$v = \tilde{\Psi} z^{\frac{\beta+\delta}{2}} \exp\left[\frac{(\rho + \tau)}{2} z^2\right] \tag{48}$$

where $\tilde{\Psi}$ obeys equation (44) with the same structure of V_{eff} but with another $\tilde{k} = \frac{(\beta-\delta)(\beta-\delta+2)}{4}$. $\tilde{\varepsilon} = (\tau - \rho)(\beta - \delta - 1) - \alpha\gamma$. It is seen that $\tilde{\varepsilon}$ can be obtained from ε by interchange between ρ and τ, β and δ. It is also seen that $\tilde{\varepsilon} - \varepsilon = 2(\rho - \tau)$.

Thus, formally, we obtain the harmonic oscillator with a barrier z^{-2} (Kratzer Hamiltonian). Let us remember, however, that the variable z in our context is complex.

It follows from (34), (43) and (48) that the functions Ψ, $\tilde{\Psi}$ obey the system of equations

$$\left[\frac{\mathrm{d}}{\mathrm{d}z} + (\tau - \rho)z + \frac{\delta - \beta}{2z}\right]\Psi = \alpha\tilde{\Psi} \tag{49}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}z} + (\rho - \tau)z + \frac{\beta - \delta}{2z}\right]\tilde{\Psi} = \gamma\Psi. \tag{50}$$

4. Conditions of quasi-exact solvability

The action of operators K_i in the subspace (27) has the same structure (8)–(10) but now the corresponding quantities A_i, B_i, C_i become 2×2 matrices:

$$K_+\vec{f}_n = C_+\vec{f}_{n+1} \tag{51}$$

$$K_-\vec{f}_n = A_-(n)\vec{f}_n + B_-(n)\vec{f}_{n-1} + C_-\vec{f}_{n+1} \tag{52}$$

$$K_0\vec{f}_n = A_0(n)\vec{f}_n + C_0\vec{f}_{n+1} \tag{53}$$

where $\vec{f}_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}$ and

$$C_+ = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad C_- = \begin{pmatrix} 2\rho^2 & 0 \\ \gamma(\tau + \rho) & 2\tau^2 \end{pmatrix} \qquad C_0 = \frac{1}{2}\begin{pmatrix} 2\rho & 0 \\ \gamma & 2\tau \end{pmatrix} \tag{54}$$

$$A_-(n) = \frac{1}{2}\begin{pmatrix} 4\beta\rho + 8n\rho + 2\rho + \alpha\gamma & 2\alpha(\tau + \rho) \\ \gamma(\beta + \delta + 2 + 4n) & 4\delta\tau + 8n\tau + 6\tau + \alpha\gamma \end{pmatrix} \tag{55}$$

$$A_0(n) = \frac{1}{4}\begin{pmatrix} 2\beta + 4n + 1 & 2\alpha \\ 0 & 2\delta + 4n + 3 \end{pmatrix} \tag{56}$$

$$B_-(n) = \frac{1}{2}\begin{pmatrix} (\beta + 2n)(\beta + 2n - 1) & \alpha(\beta + \delta + 4n) \\ 0 & (\delta + 2n + 1)(\delta + 2n) \end{pmatrix}. \tag{57}$$

In a similar way, the action of the Hamiltonian in the invariant subspace can be represented in the form

$$H\vec{f}_n = D_2\vec{f}_{n+2} + D_1(n)\vec{f}_{n+1} + \tilde{D}_0(n)\vec{f}_n + \tilde{D}_1(n)\vec{f}_{n-1} + \tilde{D}_2(n)\vec{f}_{n-2} \tag{58}$$

where the matrices D_i and $\tilde{D}_i$ have the form (12)–(16) with A_i, B_i, C_i taken from equations (54)–(57) (the order of operators is taken into account properly in this form of writing).

For the operator (1) to be QES with the invariant subspace (27), it is necessary that the matrix version of the conditions of cut off (17)–(21) be satisfied. The corresponding system of equations is too cumbersome to be listed here. It can be simplified greatly if we assume the condition $a_{--} = 0$, in which case after simple calculations we get $\tilde{D}_2 = 0$:

$$D_2 = \begin{pmatrix} \rho^2 a_{00} + 2\rho^3 a_{0-} + \frac{1}{2}\rho a_{+0} + \frac{a_{++}}{4} & 0 \\ \gamma Y & \tau^2 a_{00} + 2\tau^3 a_{0-} + \frac{1}{2}\tau a_{+0} + \frac{a_{++}}{4} \end{pmatrix} \tag{59}$$

where $Y \equiv [\frac{1}{2}(\rho+\tau)a_{00} + (\rho^2+\tau^2+\rho\tau)a_{0-} + \frac{1}{4}a_{+0}]$:

$$2\tilde{D}_1(0) = \begin{pmatrix} [\frac{a_{0-}}{4}(2\beta-3) + a_-]\beta(\beta-1) & \alpha[\frac{a_{0-}}{4}(2\beta^2+2\delta^2+2\beta\delta-3\beta-\delta) + a_-(\beta+\delta)] \\ 0 & \frac{\delta(\delta+1)}{2}[(2\delta-1)\frac{a_{0-}}{4} + a_-] \end{pmatrix} \tag{60}$$

$$D_1(N) = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \tag{61}$$

$$d_1 = \frac{a_+}{2} + 2\rho^2 a_- + \rho a_0 + \frac{a_{+0}}{8}(2\beta+4N+1) + \frac{a_{0-}}{2}[\alpha\gamma(\tau+2\rho) + \rho^2(7+6\beta+12N)] \tag{62}$$

$$+\frac{a_{00}}{4}[\rho(4\beta+8N+6)+\alpha\gamma] \tag{63}$$

$$d_2 = \alpha Y \tag{64}$$

$$d_3 = \frac{\gamma}{2}Z \qquad Z = \left[2(\tau+\rho)a_- + a_0 + \frac{a_{0-}}{2}\xi + \frac{a_{00}}{2}(\beta+\delta+4N+4)\right] \tag{65}$$

$$\xi = 12N(\rho+\tau) + \alpha\gamma + \tau(4\delta+2\beta+11) + \rho(4\beta+2\delta+9) \tag{66}$$

$$d_4 = \frac{a_+}{2} + 2\tau^2 a_- + \tau a_0 + \frac{a_{+0}}{8}(2\delta+4N+3) + \frac{a_{0-}}{2}[\alpha\gamma(\rho+2\tau) + \tau^2(12N+13+6\delta)] \tag{67}$$

$$+\frac{a_{00}}{4}[\tau(4\delta+8N+10)+\alpha\gamma]. \tag{68}$$

One can observe that

$$d_1 - d_4 = (\rho-\tau)Z + (\beta-\delta-1)Y. \tag{69}$$

The system of equations (17)–(21) with (59)–(67) taken into account looks rather cumbersome but the relation (69) simplifies the analysis significantly.

However, for a generic case $a_{--} \neq 0$ algebraic calculations are so bulky that we had to resort to using a computer.

In what follows we restrict ourselves to the Hermitian case only, which is the most interesting for physical applications. This implies that $a_{++} = a_{--}$, $a_+ = a_-$, $a_{+0} = a_{0-}$, where all coefficients are real.

The full set of nontrivial Hermitian solutions of the system (17)–(21) and their classification is given in the next section.

5. Hermitian solutions of algebraic equations

For the type of invariant subspaces (27) under consideration, we suggest below the classification of *all* QES Hermitian Hamiltonians, quadratic-linear with respect to operators K_i (i.e. those which represent even polynomials of the fourth order in terms of a, a^+).

In the tables below we list only qualitatively different cases in the following sense. If some solutions can be obtained by the limiting transition ($a_{++} \to 0$, $\gamma \to 0$, etc) from a more general case, we do not repeat them. As before, we use the abbreviation 'a.v.' for 'arbitrary value'.

5.1. $a_{++} = 0$

$\gamma \neq 0$

	δ	β	α	ρ	τ	a_-	a_{00}	a_0
1	a.v.	0	a.v.	0	$\frac{\alpha\gamma}{2(2N+1-\delta)}$	$\frac{1-2\delta}{4}a_{0-}$	$f(\tau)\,a_{0-}$	$\left[\tau g(\delta) + \frac{\delta+4N+4}{4\tau}\right]a_{0-}$
2	a.v.	0	a.v.	$\frac{\alpha\gamma}{4(N+1)}$	0	$\frac{1-2\delta}{4}a_{0-}$	$f(\rho)\,a_{0-}$	$\left[\rho g(\delta) + \frac{\delta+4N+4}{4\rho}\right]a_{0-}$
3	a.v.	1	a.v.	0	$\frac{\alpha\gamma}{2(2N+2-\delta)}$	$\frac{1-2\delta}{4}a_{0-}$	$f(\tau)\,a_{0-}$	$\left[\tau g(\delta-1) + \frac{\delta+4N+5}{4\tau}\right]a_{0-}$
4	a.v.	1	a.v.	$\frac{\alpha\gamma}{4(N+1)}$	0	$\frac{1-2\delta}{4}a_{0-}$	$f(\rho)\,a_{0-}$	$\left[\rho g(\delta-1) + \frac{\delta+4N+5}{4\rho}\right]a_{0-}$
5	0	a.v.	a.v.	0	$\frac{\alpha\gamma}{4(N+1)}$	$\frac{3-2\beta}{4}a_{0-}$	$f(\tau)\,a_{0-}$	$\left[\tau g(\beta-2) + \frac{\beta+4N+4}{4\tau}\right]a_{0-}$
6	0	a.v.	a.v.	$\frac{\alpha\gamma}{2(2N+3-\beta)}$	0	$\frac{3-2\beta}{4}a_{0-}$	$f(\rho)\,a_{0-}$	$\left[\rho g(\beta-2) + \frac{\beta+4N+4}{4\rho}\right]a_{0-}$
7	0	0	a.v.	0	a.v.	$\frac{\alpha\gamma-\tau(4N+1)}{4\tau}a_{0-}$	$f(\tau)\,a_{0-}$	$\left(-\tau-\alpha\gamma+\frac{N+1}{\tau}\right)a_{0-}$
8	0	0	a.v.	a. v.	0	$\frac{\alpha\gamma-\rho(4N+3)}{4\rho}a_{0-}$	$f(\rho)\,a_{0-}$	$\left(\rho-\alpha\gamma+\frac{N+1}{\rho}\right)a_{0-}$
9*	0	0	a.v.	0	0	$-\frac{\alpha\gamma}{2}a_{00}$	a. v.	$-2(N+1)a_{00}$
10	0	1	0	0	a. v.	$-\frac{4N+3}{4}a_{0-}$	$f(\tau)\,a_{0-}$	$\frac{4N+5}{4\tau}a_{0-}$
11	0	1	0	a.v.	0	$-\frac{4N+3}{4}a_{0-}$	$f(\rho)\,a_{0-}$	$\frac{4N+5}{4\rho}a_{0-}$
12	-1	1	a.v.	0	a.v.	$\frac{\alpha\gamma-\tau(4N+3)}{4\tau}a_{0-}$	$f(\tau)\,a_{0-}$	$\left(\tau-\alpha\gamma+\frac{N+1}{\tau}\right)a_{0-}$
13	-1	1	a.v.	a.v.	0	$\frac{\alpha\gamma-\rho(4N+1)}{4\rho}a_{0-}$	$f(\rho)\,a_{0-}$	$\left(-\rho-\alpha\gamma+\frac{N+1}{\rho}\right)a_{0-}$
14	-1	1	a.v.	0	0	$-\frac{\alpha\gamma}{2}a_{00}$	a. v.	$-2(N+1)a_{00}$
15	-1	0	0	0	a.v.	$-\frac{4N+1}{4}a_{0-}$	$f(\tau)\,a_{0-}$	$\frac{4N+3}{4\tau}a_{0-}$
16	-1	0	0	a.v.	0	$-\frac{4N+1}{4}a_{0-}$	$f(\rho)\,a_{0-}$	$\frac{4N+3}{4\rho}a_{0-}$
17	-1	a.v.	a.v.	0	$\frac{\alpha\gamma}{4(N+1)}$	$\frac{3-2\beta}{4}a_{0-}$	$f(\tau)\,a_{0-}$	$\left[\tau g(\beta-1) + \frac{\beta+4N+3}{4\tau}\right]a_{0-}$
18	-1	a.v.	a.v.	$\frac{\alpha\gamma}{2(2N+2-\beta)}$	0	$\frac{3-2\beta}{4}a_{0-}$	$f(\rho)\,a_{0-}$	$\left[\rho g(\beta-1) + \frac{\beta+4N+3}{4\rho}\right]a_{0-}$

* Note: in case 9 the coefficients $a_{+0} = a_{0-} = 0$.

where

$$g(x) \doteqdot x - 3 - 4N \tag{70}$$

and we used the definition of f according to (25).

$\gamma = 0$.

	δ	β	α	ρ	τ	a_-	a_{00}	a_0
19	a.v.	$\delta+1$	0	a.v.	ρ	$\frac{1-2\delta}{4}a_{0-}$	$f(\rho)\,a_{0-}$	$a_{0-}\left[\frac{2\delta+2N+3}{4\rho}-2\rho(N+1)\right]$
20	0	1	0	0	0	$-\frac{(4N+3)}{4}a_{0-}$	a.v.	a.v.
21	-1	0	0	0	0	$-\frac{(1+4N)}{4}a_{0-}$	0	0

5.2. $a_{++} \neq 0$

$\gamma \neq 0$. In all admissible cases (22)–(25) ρ and τ take arbitrary values:

$$a_{0-} = -\frac{1}{2\rho\,\tau}\,(\tau+\rho)\,(4\,\tau\rho+1)\,a_{+\,+} \qquad a_{00} = \frac{1}{4\rho\tau}\big((4\,\tau\rho+1)^2 + 4\big(\rho^2+\tau^2\big)\big)a_{+\,+}.$$

The rest of the relevant quantities are:

	δ	β	α	a_0	a_-
22	0	0	a.v.	$f_0(\rho,\tau)\,a_{+\,+}$	$f_-(\rho,\tau)a_{++}$
23	0	1	0	$g_0(\rho,\tau,N)a_{+\,+}$	$g_-(\rho,\tau,N)a_{+\,+}$
24	-1	1	a.v.	$f_0(\rho,\tau)\,a_{+\,+}$	$f_-(\tau,\rho)a_{++}$
25	-1	0	0	$g_0(\rho,\tau,N-\frac{1}{2})a_{+\,+}$	$g_-(\rho,\tau,N-\frac{1}{2})a_{+\,+}$

where

$$f_0(\rho,\tau) \equiv \frac{1}{2\rho\,\tau}[\big(16\tau^2\rho^2-1\big)\,(N+1) - \rho^2 + \tau^2 + \alpha\gamma\,(\tau+\rho)]$$

$$g_0(\rho,\tau,N) = \frac{(5+4\,N)}{8\rho\tau}\,\big(16\tau^2\rho^2-1\big)$$

$$g_-(\rho,\tau,N) = -\frac{(\tau+\rho)}{8\rho\tau}\,[\tau\rho(16N+28) - 4N - 3]$$

$$f_-(\rho,\tau) \equiv -\frac{1}{8\rho\,\tau}[\tau^2\rho(16N+28) + \tau\,\rho^2(20+16N) + \alpha\gamma(4\rho\tau+1) - \rho(4N+3) - \tau(1+4\,N)]$$

$\gamma = 0$. Now $\rho = \tau$:

	δ	β	α	a_{00}	a_{0-}	a_0
26	0	0	a.v.	$f_{00}(\tau)a_{+\,+}$	$2f(\tau)a_{+\,+}$	$-f^2(\tau)a_{++} + f(\tau)a_-$
27	0	1	0	$f(\tau)a_{0-} + f(2\tau^2)a_{++}$	a.v.	$f(\tau)a_- - g_0(\tau,\tau,N)a_{++}$ $+f_1(\tau, N+\frac{1}{2})a_{0-}$
28	-1	1	a.v.	$f_{00}(\tau)a_{+\,+}$	$2f(\tau)a_{++}$	$-f^2(\tau)a_{++} + f(\tau)a_-$
29	-1	0	0	$f(\tau)a_{0-} + f(2\tau^2)a_{++}$	a.v.	$f(\tau)a_- - g_0(\tau,\tau,N-\frac{1}{2})a_{++}$ $+f_1(\tau,N)a_{0-}$

where

$$f_{00}(\tau) = \frac{1}{4\tau^2}\left(1 + 16\tau^4 + 16\,\tau^2\right)$$

and the function f_1 is defined according to (25).

6. Explicit examples of invariant subspaces

In this section we briefly list the explicit form of solutions for $u(z)$ and $v(z)$.

(1) Case 9: $\beta = \delta = \rho = \tau = 0$,

(a) $\gamma = -\alpha = -\omega \neq 0$.
Then it follows directly from (34) that the functions u, v can be chosen as $u = \cos\omega z$, $v = \sin\omega z$. After simple calculations one finds that, apart from the Hermitian QES Hamiltonian, there also exists the non-Hermitian QES operator:

$$H = a_{00}K_0^2 + a_{0-}K_0K_- + \left[\frac{a_{0-}}{2}\omega^2 - 2a_{00}(N+1)\right]K_0 + a_-K_- + \frac{a_{00}}{2}\omega^2 K_+. \qquad (71)$$

(b) In a similar way, we obtain for $\gamma = \alpha = \omega$ that $u = \cosh(\omega z)$, $v = \sinh\omega z$, and the operator H is obtained by the replacement $\omega^2 \to -\omega^2$ in the expression (71).

(2) $\rho = \tau = 0, \delta = -1 - n, \beta = n\ (n = 0, 1\ldots), \alpha = -1, \gamma = 1$.
Then we have the following solutions of (34): $u = J_n(z)$, $v(z) = J_{n+1}(z)$ (Bessel functions):

$$a_- = \frac{(3+2n)}{4}a_{0-} \qquad a_{+0} = 0 = a_{++} \qquad a_+ = \frac{a_{00}}{2} \qquad a_0 = \frac{a_{0-}}{2} - \frac{a_{00}}{2}(4N+3). \qquad (72)$$

The effective Hamiltonian is non-Hermitian.

(3) Consider case 8 of Hermitian Hamiltonians (case 7 can be considered in a similar manner): $\beta = \delta = 0 = \tau$, then $k = 0 = \tilde{k}$ and formally we have in the z representation the wavefunction that looks the way that a pure harmonic oscillator would look in the coordinate representation. Equations (49) and (50) take the form

$$b\Psi = \frac{\alpha}{\sqrt{2\rho}}\tilde{\Psi} \qquad (73)$$

$$b^+\tilde{\Psi} = \frac{\gamma}{\sqrt{2\rho}}\Psi \qquad (74)$$

where $b = \frac{1}{\sqrt{2\rho}}[\frac{\mathrm{d}}{\mathrm{d}z} - \rho z]$, $b^+ = \frac{1}{\sqrt{2\rho}}[\frac{\mathrm{d}}{\mathrm{d}z} + \rho z]$. It is obvious that $[b, b^+] = 1$.

The frequency is equal to $\omega = 2\rho$, $\varepsilon = -\frac{\omega}{2} - \alpha\gamma$, $\tilde{\varepsilon} - \varepsilon = 2\rho$. Let also $\varepsilon = \varepsilon_n \equiv \omega(n+1/2), \alpha\gamma = -\omega(n+1)$. Then we have the eigenvalue and $\tilde{\varepsilon} = \varepsilon_{n+1}$. Thus, our subspace is span$\{\Psi_n z^{2n}\exp[\left(\frac{\rho}{2}\right)z^2], \Psi_{n+1}z^{2n+1}\exp[\left(\frac{\rho}{2}\right)z^2]\}$, where Ψ_n is the wavefunction of the nth level of the harmonic oscillator, $\Psi_n = \exp(-\frac{1}{2}\rho z^2)H_n(z\sqrt{\rho})$, and H_n is the Hermite polynomial. Here $\beta = 0, 1, \ldots$.

We would like to stress that our system represents an anharmonic (not harmonic!) Bose oscillator. The functions, which have the same form as those of a harmonic oscillator, appear in this context in the coherent state representation (not in the coordinate one, as would be the case for the usual harmonic oscillator) and represent auxiliary quantities.

7. Generalizations

In this section we describe briefly, on the basis of the suggested approach, some possible ways of generating new invariant subspaces, suitable for constructing QES Bose Hamiltonians. As the method of construction is the same as was used above, we only dwell upon the structure of subspaces.

(1) Let us introduce the quantities

$$f_n^1 = z^{2n}u^2 \qquad f_n^2 = z^{2n+1}uv \qquad f_n^3 = z^{2n}v^2 \tag{75}$$

and consider the subspace $F_N = \operatorname{span}\{f_n^1,\ f_n^2,\ f_n^3\}$, $n = 0, 1, \ldots, N$; $N = 1, 2, \ldots$. Let $\vec{f}_n \equiv \begin{pmatrix} f_n^1 \\ f_n^2 \\ f_n^3 \end{pmatrix}$. Then (51)–(53) take place, where, however, the corresponding matrices now have dimension 3×3:

$$A_0(n) = \begin{pmatrix} \xi + \frac{1}{4} & \alpha & 0 \\ 0 & \frac{2\xi+2\eta+3}{4} & 0 \\ 0 & \gamma & \eta + \frac{1}{4} \end{pmatrix} \qquad C_0 = \begin{pmatrix} 2\rho & 0 & 0 \\ \frac{\gamma}{2} & \omega & \frac{\alpha}{2} \\ 0 & 0 & 2\tau \end{pmatrix} \qquad C_+ = \tfrac{1}{2}I \tag{76}$$

I is a unit matrix,

$$A_-(n) = \begin{pmatrix} 2\rho\,(4\xi+1) + \alpha\gamma & 2\alpha\,(\omega+2\rho) & \alpha^2 \\ \frac{\gamma}{2}\,(2+3\xi+\eta) & \omega\,(2\xi+2\eta+3) + 2\alpha\gamma & \frac{\alpha}{2}\,(2+\xi+3\eta) \\ \gamma^2 & 2\gamma\,(\omega+2\tau) & 2\tau\,(1+4\eta) + \alpha\gamma \end{pmatrix} \tag{77}$$

$$B_-(n) = \begin{pmatrix} \xi\,(2\xi-1) & \alpha\,(3\xi+\eta) & 0 \\ 0 & \frac{1}{2}\,(\eta+\xi+1)\,(\eta+\xi) & 0 \\ 0 & \gamma\,(3\eta+\xi) & \eta\,(2\eta-1) \end{pmatrix} \tag{78}$$

$$C_-(n) = \begin{pmatrix} 8\rho^2 & 0 & 0 \\ \gamma\,(\omega+2\rho) & 2\omega^2 & \alpha\,(2\tau+\omega) \\ 0 & 0 & 8\tau^2 \end{pmatrix} \tag{79}$$

$\xi = \beta + n$, $\eta = n + \delta$, $\omega = \rho + \tau$.

(2) Further generalization consists in considering

$$f_n^1 = z^{2n}u\tilde{u} \qquad f_n^2 = z^{2n+1}\tilde{u}v \qquad f_n^3 = z^{2n+1}u\tilde{v} \qquad f_n^4 = z^{2n}v\tilde{v}. \tag{80}$$

Here $\tilde{u}$, $\tilde{v}$ refer to the functions obeying equations (34) with parameters $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\rho}, \tilde{\tau}$. Then we can construct $\operatorname{span}\{f_n^1,\ f_n^2,\ f_n^3, f_n^4\}$.

(3) Consider $\operatorname{span}\{z^{2n+\xi(k)}u^{M-k}v^k\}$, $\xi(k) = 1$, if k is odd, and $\xi = 0$ if k is even. Here $k = 0, 1, 2 \ldots M$, $n = 0, 1 \ldots N$.

The functions $\{u^i\}$ that appear in subspaces (1)–(3) obey the system of equations of the type $\frac{\mathrm{d}u^i}{\mathrm{d}z} = C^i_j\,(z)\,u^j$ that contains, as a particular case, equations (34).

8. Summary

Let us summarize the basic features of our approach to constructing invariant subspaces for Bose Hamiltonians. We consider systems whose Hamiltonian can be expressed in terms of the generators of K_i (2). Further, we (1) use the coherent state representation in which all Bose operators become differential ones, (2) split the space of states of a harmonic oscillator into even and odd states, (3) deform each of the two pieces by introducing, as a factor, an additional unknown function which is different for each piece, (4) demand that both these functions obey a coupled system of differential equations such that the action of K_i convert each of the basis vectors into a linear combination of vectors of the same type, (5) select coefficients, for which K_i enter the Hamiltonian H, to ensure the cut off in the space of basis functions. In a sense, we introduce a kind of additional degree of freedom—effective 'spin' s. Then the kinds of subspaces considered in our paper can be assigned the values $s = 0$ (section 2), $1/2$ (section 3), 1 (section 7), etc. However, we want to stress that this 'spin', in contrast to the matrix QES models [20–23], does not appear in the Hamiltonian and serves only to describe the structure of the solutions.

It is worth stressing that it is just H itself that is QES, whereas the generators K_i themselves, with the help of which H is built up, are, in general, not. This is in sharp contrast to the 'usual' QES Hamiltonians in quantum mechanics. Let us remind ourselves that in the latter case H can be expressed in terms of operators of an effective spin S_i that realize the $sl(2, R)$ algebra, each of them possessing a finite-dimensional subspace.

The essential ingredient of our approach is using the coherent state representation. Formally, formulae (4) look like those in the coordinate-momentum representation. However, for our Hamiltonian one should check carefully the normalizability of solutions that implies integration over the whole complex plane in a scalar product and impose constraints on admissible values of the parameters. Another constraint stems from the demand of analyticity.

We want to point out that non-Hermitian QES operators can also be of interest. They may be used, as auxiliary quantities, in physical applications for finding spectra of Hermite Hamiltonians. For instance, they may appear in mappings like $HL = LH'$, where H is Hermitian. Knowing the spectrum of H' or its part due to its quasi-exact solvability, one can restore a part of the spectrum of a physical Hamiltonian H. Apart from this, the approach considered in the present paper opens up a way to the search and classification of linear differential operators with different invariant subspaces. This would enable one to generalize or extend the results obtained for such subspaces with a basis of monomials [24, 25]. In particular, for the case (72) we obtained solutions in the form of combinations of Bessel functions and monomials.

In this paper we restricted ourselves to one-particle systems but the suggested approach is obviously extendable to many-particle Hamiltonians. It also enables one to generate new QES Bose Hamiltonians by choosing another kind of functions A, B, C, D in equations (28) and (29).

The suggested approach can be useful in the problems of solid state physics when interaction between phonons is essential, quantum optics, theory of molecules, etc. Especially important in this context is the fact that our approach is extendable to many-particle systems. Concrete elaboration and applications of the obtained results deserve special treatment.

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